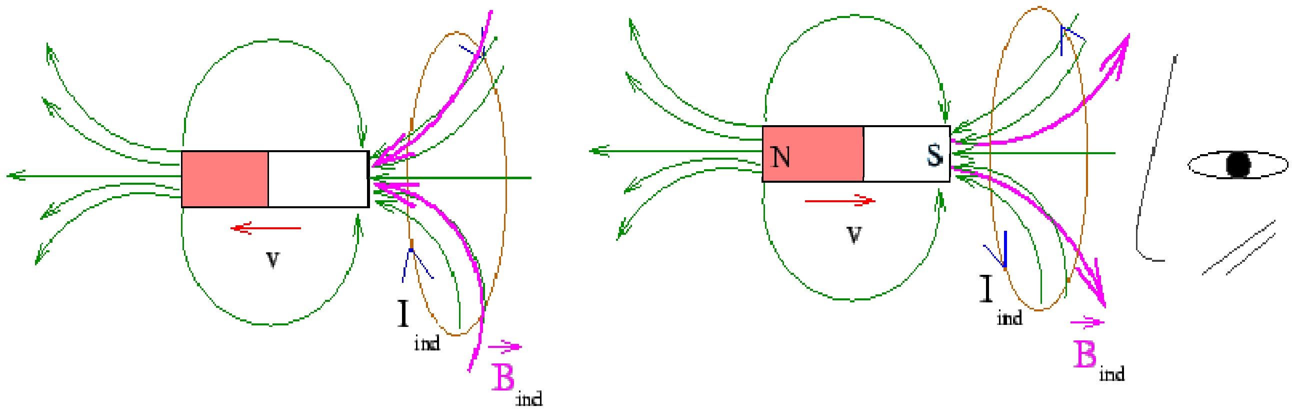
2nd _sem_CC3T, Mahadeb Pal, Assistant professor of physics.
Electromagnetic Induction 1 :

We have seen that studies made by Oersted, Biot-Savart and Ampere showed that an electric current produces a magnetic field. Michael Faraday wanted to explore if this phenomenon is reversible in the sense whether a magnetic field could be source for a current in a conductor. However, no current was found when a conductor was placed in a magnetic field. Faraday and (Joseph) Henry, however, found that if a current loop was placed in a time varying magnetic field or if there was a relative motion between a magnet and the loop a transient current was established in the conducting loop. They concluded that the source of the electromotive force driving the current in the conductor is not the magnetic field but the changing magnetic flux associated with the loop. The change in flux could be effected by (i) a time varying magnetic field or by (ii) motion of the conductor in a magnetic field or (iii) by a combined action of both of these. The discovery is a spectacular milestone in the sense that it led to important developments in Electrical engineering like invention of transformer, alternator and generator.
Shortly after Faraday's discovery, Heinrich Lenz found that the direction of the induced current is such that it opposes the very cause that produced the induced current (i.e. the magnetic field associated with the induced current opposes the change in the magnetic flux which caused the induced current in the first place). Lenz's law is illustrated in the following.



In the figures the loops are perpendicular to the plane of the page. The direction of induced current is as seen towards the loop from the right. Note that the magnetic field set up by the induced current tends to increase the flux in the case where the magnet is moving away from the loop and tends to decrease it in the case where it is moving towards the loop,
Mathematically, Faraday's law is stated thus : the electromotive force is proportional to the rate of change of magnetic flux. In SI units, the constant of proportionality is unity.

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

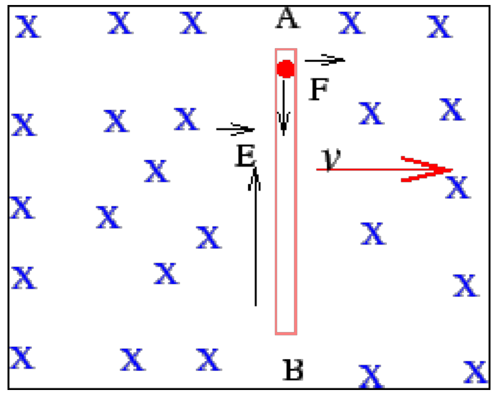
where Φ is the flux associated with the circuit and the minus sign is a reminder of the direction of the current as given by Lenz's law. If the loop contains N turns, the equation becomes

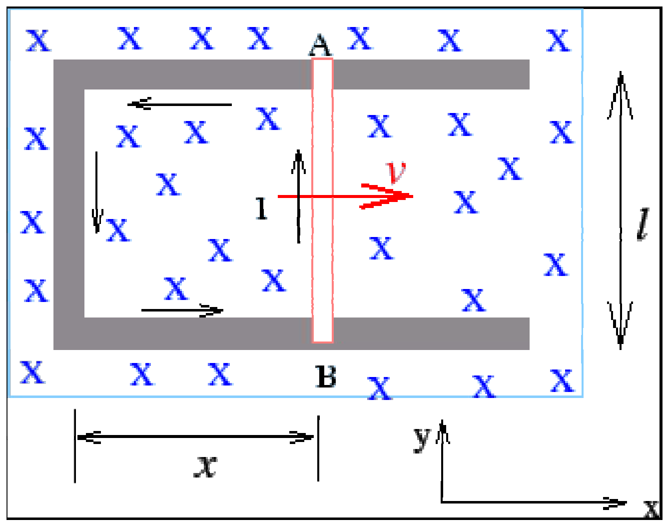
$$\mathcal{E} = -N\frac{d\Phi}{dt}$$

Though the flux is a scalar, one can fix its sign by considering the sign of the area vector which is fixed by the usual right hand rule. The dot product of $\vec{B}$ and $d\vec{S}$ then has a sign.

Motional Emf :

Consider a straight conductor AB moving along the positive x-direction with a uniform speed v. The region is in a uniform magnetic field pointing into the plane of the page, i.e. in $-z$ direction.





The fixed positive ions in the conductor are immobile. However, the negatively charged electrons experience a Lorentz force $-|e|\vec{v}\times\vec{B} = -|e|vB\hat{j}$, i.e. a force along the $-y$ direction. This pushes the electrons from the end A to the end B, making the former positive with respect to the latter. Thus an induced electric field is established in the conductor along the positive y direction. The acceleration of electrons would stop when the electric field is built to a strength which is strong enough to annul the magnetic force. This electric field $\vec{E} = \vec{v}\times\vec{B}$ is the origin of what is known as *motional emf* . The motion of charges finally stops due to the resistance of the conductor

If the conductor slides along a stationary U- shaped conductor, the electrons find a path and a current is established in the circuit. The moving conductor thereby becomes a seat of the motional emf. We may calculate the emf either by considering the work that an external agency has to do to keep the sliding conductor move with a uniform velocity or by direct application of Faraday's law.

If the induced current is I, a force IlB acts on the wire in the negative x direction. In order to maintain the uniform velocity, an external agent has to exert an equal and opposite force on the sliding conductor. Since the distance moved in time dt is vdt, the work done by the external agency is

$$dW = IlB.vdt = Blv(dq)$$

where $dq = Idt$ is the amount of charge moved by the seat of emf along the direction of the current. The emf is an electric potential difference.

Thus, the emf is equivalently the work done in moving a unit charge. Thus,

$$\mathcal{E} = \frac{dW}{dq} = Blv$$

This emf corresponds to the potential difference beteen the ends A and B.
An alternate derivation of the above is to consier the flux linked with closed circuit. Taking the origin at the extreme left end of the circuit, the area of the circuit in the magnetic field is lx where x is the distance of the sliding rod from the fixed end. The flux linked with the circuit is, therefore, $\Phi = Blx$. The rate at which flux changes is therefore given by

$$\mathcal{E} = \frac{d\Phi}{dt} = Bl\frac{dx}{dt} = Blv$$

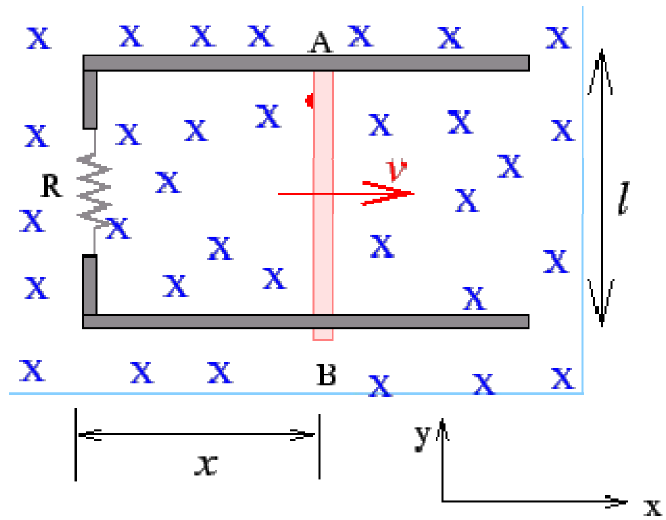
If $\vec{B}$ is not perpendicular to the plane of the circuit, we will need to take the perpendicular component of $\vec{B}$ in the above formula.
Note that, in the illustration above, the magnetic flux linked with the circuit is increasing with time in the negative z direction. The direction of the induced current is, therefore, such that the magnetic field due to the current is along the positve z-direction, which will oppose increase of flux. Hence the current, as seen from above, is in the anticlockwise direction.

Example 1

In the above circuit if the part of the fixed rails parallel to the sliding conductor has a resistance R and the rest of the circuit may be considered resistanceless, obtain an expression for the velocity of the conductor at time t, assuming that it starts with an initial velocity v_0 at $t = 0$. Explain the change in the kinetic energy of the sliding rod.

Solution :

Since the emf is $\mathcal{E} = Blv$, the current flowing through the sliding conductor is $I = Blv/R$.



The force acting on the sliding conductor is $F = I\vec{L} \times \vec{B} = -\frac{B^2 l^2 v}{R}\hat{i}$

The force is a retarding one, slowing down the conductor in accordance with Lenz's law.

(This is the amount of force an external agency must apply on the sliding conductor in the positive x-direction to keep the rod moving with a constant speed.) Thus the equation of motion of the conductor is

$$m\frac{dv}{dt} = -\frac{B^2 l^2 v}{R}$$

The equation may be solved by separating the variables and integrating from time $t = 0$ to t. We get

$$\int_{v_0}^{v} \frac{dv}{v} = -\int_0^t \frac{B^2 l^2}{mR} dt$$

which gives

$$\ln\frac{v}{v_0} = -\frac{B^2 l^2}{mR} t$$

i.e.

$$v = v_0 \exp\left(-\frac{B^2 l^2}{mR} t\right)$$

We can calculate the power dissipated in the circuit in this time by using $dW/dt = I^2 R = B^2 l^2 v^2/R$, so that

$$\begin{aligned} W &= \frac{B^2 l^2}{R}\int_0^t v^2 dt \\ &= \frac{B^2 l^2}{R} v_0^2 \int_0^t e^{-\frac{2B^2 l^2}{mR}t} dt \\ &= (mv_0^2/2)\left[1 - e^{-2\frac{B^2 l^2}{mR}t}\right] \end{aligned}$$

which is precisely the change in the kinetic energy of the conductor $(mv_0^2/2) - (mv^2/2)$

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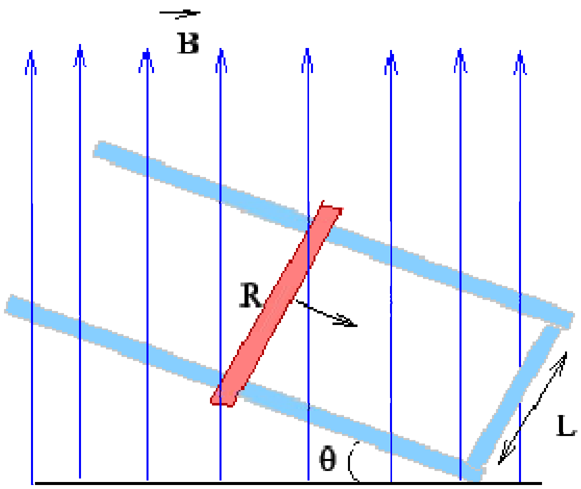
Exercise 1

A pair of parallel conducting rails are inclined at an angle θ to the horizontal. The rails are connected to each other at the ground by a conducting strip. A conductor of resistance R, oriented parallel to the strip can slide down the incline along the rails. The resistances of the rails and the strip are negligible.

A uniform magnetic field $\vec{B}$ exists in the vertical direction. The slider is released at some height. Show that the slider attains a terminal speed given by

$$v = \frac{mgR\sin\theta}{B^2L^2\cos^2\theta}$$

where L is the distance between the rails.



Exercise 2

In the above problem, show that after attaining the terminal velocity, the change in the potential energy of the slider is equal to the Joule heat produced in the slider.

Hint for Solution :

In the preceding exercise, show that the current is given by $BLv\cos\theta/R$. The rate of Joule heat is I^2R. The amount of heat produced in time t is I^2Rt. In time t, the slider moves through a distance vt. The change in the potential energy is $mgvt\sin\theta$.

Example 2

A conductor AB is moving with a speed v parallel to a long straight wire carrying a current I. What is the potential difference between the ends A and B ?

Solution :

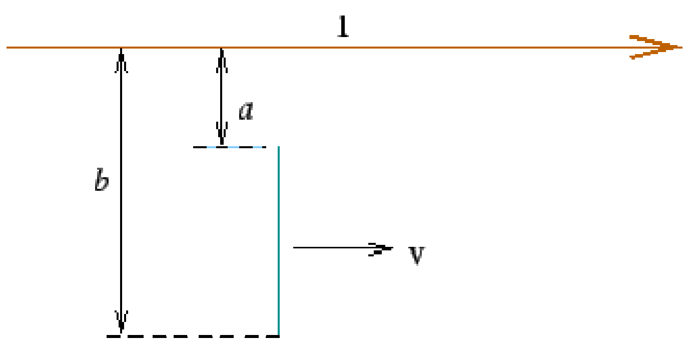
From the discussion above, the electric field acting at any point on the moving conductor is Lorentz force per unit charge, i.e., $\vec{v}\times\vec{B}$. Since the field is not constant along AB, the emf is obtained by integrating the electric field

along AB.
The field at an element of width dx at a distance x from the wire is. Since **B** and ***V*** are perpendicular, $\mu_0 I/2\pi x$

$$\mathcal{E} = \int_a^b \frac{mu_0 I v}{2\pi}\frac{dx}{x} = \frac{\mu_0 I v}{2\pi}\ln(a/b)$$

One can obtain the same expression by a direct application of Faraday's law. It is not necessary to have a physical circuit to calculate the potential difference. One can imagine the conductor AB to be a rod sliding along a rail. As the rod moves, the area of the closed region ACDB increases. We cansider an area element of width dx at a distance x from the wire. Since $\vec{B}$ and the area vector are parallel,

$$d\Phi = \frac{\mu_0 I}{2\pi} L \frac{dx}{x}$$



Integrating, $$\Phi = \frac{\mu_0 I}{2\pi} L \ln(b/a)$$

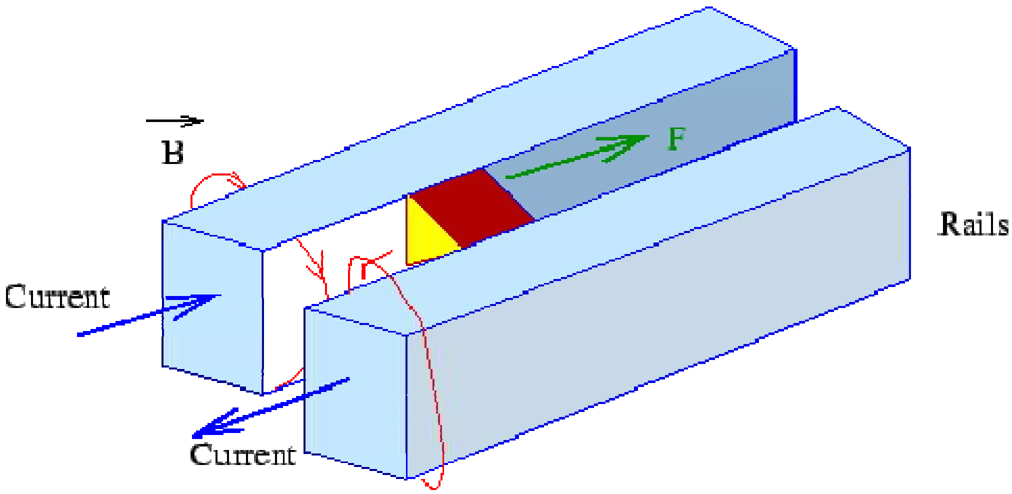
Thus

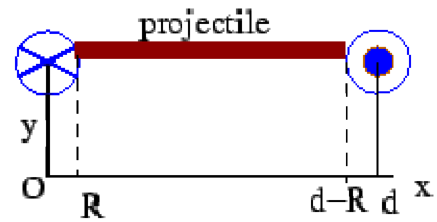
$$\mathcal{E} = -\frac{d\Phi}{dt} = \frac{\mu_0 I}{2\pi}\ln(a/b)\frac{dL}{dt} = \frac{\mu_0 I}{2\pi} v \ln(a/b)$$

since the rate of increase of the length of the rectangle is $dL/dt = v$.

Example 3

A rail gun is essentially a linear electromagnetic accelerator which uses Lorentz force to propel an armature (connected to a projectile) between two parallel rails carrying high current. The current enters through one of the rails and returns through the other rail. The cuurent in the rails produce magnetic field in the region between the rails. The armature is subject to a force parallel to the rails.
The separation between the rails is small compared to the length of the rails so that the magnetic field due to the rails may be approximated as due to infinitely long wires carrying current. However, because the armature provides a path for the current, there is no current in the rails beyond the armature. The magnetic field is, therefore, equal to that due to semi-infinite wire (i.e. reduced by a factor of 2 from that of an infinite wire).





In the figure shown, the radii of cross section of the rails are R each and the separation between them is d. The fields due to both the wires are in the $-y$ direction, so that the force $I\vec{dl} \times \vec{B}$ acts along the z direction, parallel to the rails. The field at a distance x from the origin shown is $\vec{B} = -\frac{\mu_0 I}{4\pi}\left[\frac{1}{x} + \frac{1}{d-x}\right]$

The force on the armature is obtained by integrating the force on the element dx of the armature

$$
\begin{aligned}
d\vec{F} &= I\vec{dl} \times \vec{B} \\
&= I^2\frac{\mu_0 I}{4\pi}dx\left[\frac{1}{x} + \frac{1}{d-x}\right]\hat{\imath} \times (-\hat{\jmath}) \\
&= -I^2\frac{\mu_0 I}{4\pi}\hat{k}dx\left[\frac{1}{x} + \frac{1}{d-x}\right]
\end{aligned}
$$

The force on the armature is

$$
\begin{aligned}
\vec{F} &= -I^2\frac{\mu_0 I}{4\pi}\hat{k}\int_R^{d-R}\left[\frac{1}{x} + \frac{1}{d-x}\right]dx \\
&= -I^2\frac{\mu_0}{2\pi}\hat{k}\left[\ln(d-R) - \ln(R)\right]
\end{aligned}
$$

A typical rail gun design to achieve high acceleration of a projectile, a pulse current in excess of 100 kAmp is used. With a rail separation of $d = 5$cm and $R = 1$cm, we have $F = 2772$m/s ^{2}. For a projectile mass of 20 gms. the acceleration produced is in excess of $14,000g$!

Example 4

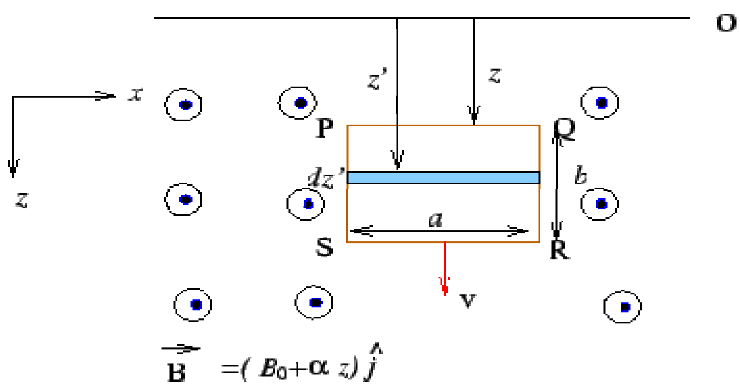
A rectangular conducting loop of length a and width b, having a resistance R falls under gravity with its smaller side remaining vertical during the fall. A non-uniform magnetic field exists, directed horizontally and perpendicular to the plane of the loop. The magnitude of the magnetic field increases linearly with distance $B(z) = B_0 + \alpha z$, where z is the distance of the top of the loop from the position from which the loop falls. Obtain an expression for the velocity of the loop with distance of fall and find the terminal velocity.

Solution :

Let the axes be as shown, with the y-axis out of the plane of the paper. The induced current in the loop is clockwise so as to oppose the flux increase as the loop falls. The velocity of the loop is $v = dz/dt$.

The flux through a strip of width dz' at position z' is $(B_0 + \alpha z')adz'$. The net flux through the loop is

$$\begin{aligned}\Phi &= \int_z^{z+b} (B_0 + \alpha z')adz' \\ &= B_0 ab + \frac{\alpha a}{2}(2zb + b^2)\end{aligned}$$



The induced emf is $\mathcal{E} = \frac{d\Phi}{dt} = \alpha ab\frac{dz}{dt} = \alpha abv$

The induced current is $I = \alpha abv/R$. When the loop falls, the sides of the loop are subject to magnetic forces $I\vec{l} \times \vec{B}$.

The forces on the sides QR and SP cancel. The magnetic force on the sides PQ and RS are

$$\begin{aligned}\vec{F}_m &= IaB(z)\hat{k} - IaB(z+b)\hat{k} \\ &= Ia(B_0 + \alpha z)\hat{k} - Ia\left[B_0 + \alpha(z+b)\right]\hat{k} \\ &= -Iab\alpha\hat{k} \\ &= -(\alpha ab)^2\frac{v}{R}\hat{k}\end{aligned}$$

The equation of the motion of the loop having a mass m is $m\frac{dv}{dt} = mg - \frac{(\alpha ab)^2}{R}v$

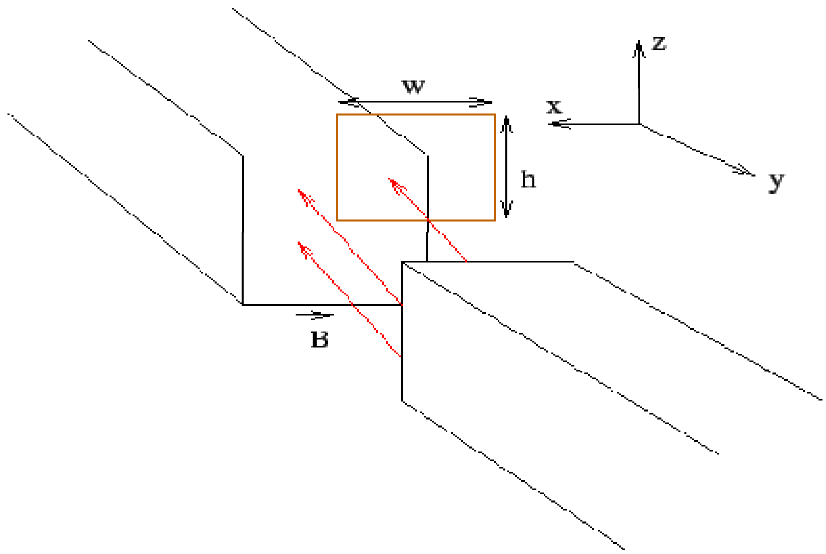
which has a solution

$$v = \frac{mgR}{(\alpha ab)^2}\left[1 - \exp(-\frac{(\alpha ab)^2}{R}t\right]$$

The terminal velocity is $v_\infty = mgR/(\alpha ab)^2$, which can also be directly obtained by equating F_m with mg.

Exercise 3

A rectangular loop of width w and height h falls under gravity into a region of constant magnetic field B. The loop has a mass m and resistance R. The magnetic field, which remains perpendicular to the plane of the loop, is constant within the pole pieces and is zero outside.



(a) Find the current in the loop when the speed of the loop is v and (i) it is partly inside and partly outside the field region (ii) it is wholely inside the filed.

(b) Find the force acting on the loop in both the cases above.

(c) Determine the terminal velocity of the loop.

Assume that the pole pieces are much deeper than the height h.

(**Answer** : (a) (i) Bwv/R counter-clockwise (ii) zero. (b) B^2w^2v/R acts only on the lower edge. (c) terminal speed in case (i) is $v_0 = mgR/B^2w^2$. Once wholely inside, the speed increases with time as $v = v_0 + gt$.)

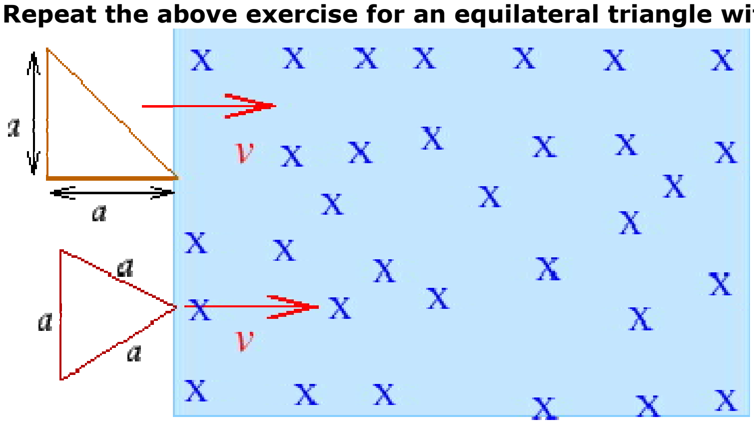
Exercise 4

A triangular current loop in the shape of an right isoceles triangle of base $a = 1$ m enters a region of constant magnetic field with a uniform speed $v = 2$m/s. At $t = 0$ a corner of the base is at the edge of the field region (top of the figure). Find the emf at time (i) $t = 0.3$s and (ii) $t = 1$s.

(Ans. (i) 0.6 volts (ii) zero.)

Exercise 5

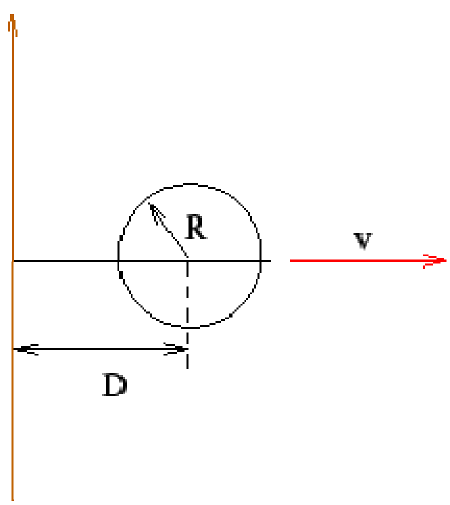
Repeat the above exercise for an equilateral triangle with side $a = 1$ m.



(Answer : 0.7 volts)

Exercise 6

(Mathematically difficult problem) : A circular loop of radius R lies in the same plane as a long straight conductor carrying a current I. The centre of the loop is at a distance D from the wire ($D > R$). The loop moves perpendicular to the wire with a uniform speed v. Calculate the motional emf developed.



(Hint : Consider the field at an element of area $r\,dr\,d\theta$. The distance of the element from the wire is $D + r\cos\theta$. Calculate the flux by integrating from $r = 0$ to R and from $\theta = 0$ to 2π. Use :

$$\int_0^{2\pi} \frac{d\theta}{D + r\cos\theta} = \frac{2\pi}{\sqrt{D^2 - r^2}}$$

(Answer $\mu_0 I v(D/\sqrt{D^2 - R^2} - 1)$)

Time Varying Field

Even where there is no relative motion between an observer and a conductor, an emf (and consequently an induced current for a closed conducting loop) may be induced if the magnetic field itself is varying with time as flux change may be effected by change in magnetic field with time. In effect it implies that a changing magnetic field is equivalent to an electric field in which an electric charge at rest experiences a force.

Consider, for example, a magnetic field $\vec{B}$ whose direction is out of the page but whose magnitude varies with time. The magnetic field fills a cylindrical region of space of radius R. Let the magnetic field be time varying and be given by

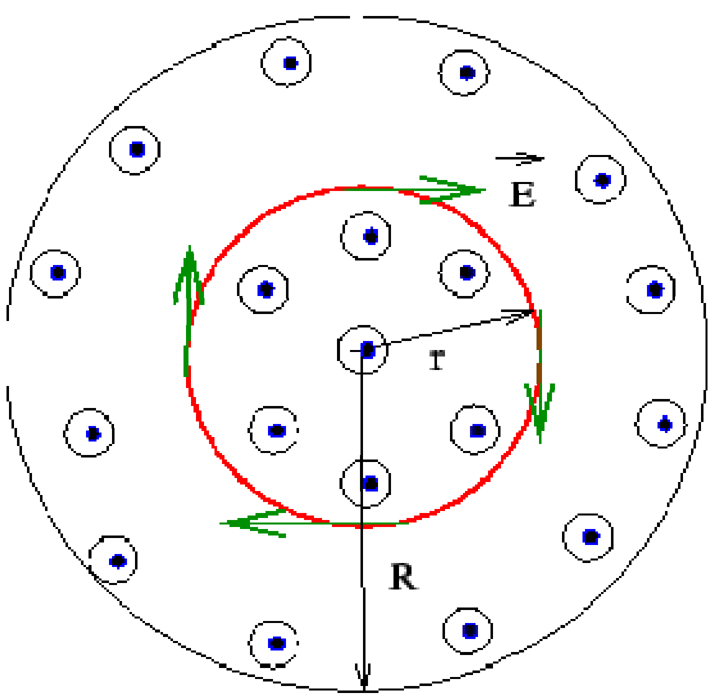
$$B = \begin{cases} B_0 \sin \omega t & \text{for } r \leq R \\ 0 & \text{for } r > R \end{cases}$$

Since $\vec{B}$ does not depend on the axial coordinate z as well as the azimuthal angle ϕ , the electric field is also independent of these quantities. Consider a coaxial circular path of radius $r \leq R$ which encloses a time varying flux. By symmetry of the problem, the electric field at every point of the cicular path must have the same magnitude E and must be tangential to the circle.

Thus the emf is given by $\mathcal{E} = \int \vec{E} \cdot \vec{dl} = 2\pi r E$

By Faraday's law

$$\begin{aligned} \mathcal{E} &= -\frac{d\Phi}{dt} = -\frac{d}{dt}[\pi r^2 B(t)] \\ &= -\pi r^2 \frac{dB(t)}{dt} \end{aligned}$$



Equating these, we get for $r \leq R$,$E = -\frac{r}{2}\frac{dB}{dt} = -\frac{B_0\omega}{2} r \cos\omega t$

For $r > R$, the flux is $\Phi = \pi R^2 B(t)$, so that $\mathcal{E} = -\frac{d\Phi}{dt} = -\pi R^2 \frac{dB}{dt}$

and the electric field foir $r > R$ is $E = -\frac{R^2}{2r}\frac{dB}{dt} = -\frac{B\omega R^2}{2r}\cos\omega t$

Exercise 1

A conducting circle having a radius R_0 at time $t = 0$ is in a constant magnetic field B perpendicular to its plane. The circle expands with time with its radius becoming $R = R_0(1 + \alpha t^2)$ at time t. Calculate the emf developed in the circle.

(Ans. $-4\pi R_0^2 \alpha t(1 + \alpha t^2) B$)